Classification of upper motives of algebraic groups of inner type A_n
Classification des motifs supérieurs des groupes algébriques intérieurs de type A_n.

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Résumé

Soient A, A' deux algèbres centrales simples sur un corps F et $\mathbb{F}$ un corps fini de caractéristique p. Nous prouvons que les facteurs directs indécomposables supérieurs des motifs de deux variétés anisotropes de drapeaux d'idéaux à droite $X(d_1, ..., d_k; A)$ et $X(d'_1, ..., d'_s; A')$ à coefficients dans $\mathbb{F}$ sont isomorphes si et seulement si les valuations p-adiques de $\mathrm{pgcd}(d_1, ..., d_k)$ et $\mathrm{pgcd}(d'_1, .., d'_s)$ sont égales et les classes des composantes p-primaires A_p et A'_p de A et A' engendrent le même sous-groupe dans le groupe de Brauer de F. Ce résultat mène à une surprenante dichotomie entre les motifs supérieurs des groupes algébriques absolument simples, adjoints et intérieurs de type A_n.

Abstract

Let A, A' be two central simple algebras over a field F and $\mathbb{F}$ be a finite field of characteristic p. We prove that the upper indecomposable direct summands of the motives of two anisotropic varieties of flags of right ideals $X(d_1, ..., d_k; A)$ and $X(d'_1, ..., d'_s; A')$ with coefficients in $\mathbb{F}$ are isomorphic if and only if the p-adic valuations of $\gcd(d_1, ..., d_k)$ and $\gcd(d'_1, .., d'_s)$ are equal and the classes of the p-primary components A_p and A'_p of A and A' generate the same group in the Brauer group of F. This result leads to a surprising dichotomy between upper motives of absolutely simple adjoint algebraic groups of inner type A_n.

1 Introduction

Throughout this note p will be a prime and $\mathbb{F}$ will be a finite field of characteristic p. Let F be a field, F-$\mathfrak{alg}$ be the category of commutative F-algebras and $\mathrm{CM}(F;\mathbb{F})$ be the category of Grothendieck Chow motives with coefficients in $\mathbb{F}$. Motivic properties of projective homogeneous F-varieties and their relations with classical discrete invariants have been intensively studied recently (see for example [7], [11], [12], [13], [14], [15]). In this article we deal with the particular case of projective homogeneous F-varieties under the action of an absolutely simple affine adjoint algebraic group of inner type A_n. More precisely we prove the following result:

Theorem 1. Let A and A' be two central simple F-algebras. The upper indecomposable direct summands of the motives of two anisotropic varieties of flags of right ideals $X(d_1,...,d_k;A)$ and $X(d'_1,...,d'_s;A')$ in $\mathrm{CM}(F;\mathbb{F})$ are isomorphic if and only if $v_p(\gcd(d_1,...,d_k)) = v_p(\gcd(d'_1,..,d'_s))$ and the p-primary components A_p and A'_p of A and A' generate the same subgroup of $\mathrm{Br}(F)$.

In §1 we recall classical discrete invariants and varieties associated to central simple F-algebras, while §2 is devoted to the theory of upper motives. Finally we prove theorem 1 in §3, using an index reduction formula due to Merkurjev, Panin and Wadsworth and the theory of upper motives. Theorem 1 gives a purely algebraic criterion to compare upper direct summands of varieties of flags of ideals, and leads to a quite unexpected dichotomy between upper motives of absolutely simple adjoint algebraic groups of inner type A_n.

2 Generalities on central simple algebras

Our reference for classical notions on central simple F-algebras is [9]. A finite-dimensional F-algebra A is a central simple F-algebra if its center $Z(A)$ is equal to F and if A has no non-trivial two-sided ideals. The square root of the F-dimension of A is the *degree* of A, denoted by $\deg(A)$. Two central simple F-algebras A and B are *Brauer-equivalent* if $M_n(A)$ and $M_m(B)$ are isomorphic for some integers n and m, and the *Schur index* $\mathrm{ind}(A)$ of a central simple F-algebra A is the degree of the (uniquely determined up to isomorphism) central division F-algebra Brauer-equivalent to A. The tensor product endows the set $\mathrm{Br}(F)$ of equivalence classes of central simple F-algebras under the Brauer equivalence with a structure of a torsion abelian group. The exponent of A, denoted by $\exp(A)$, is the order of the class of A in $\mathrm{Br}(F)$ and divides $\mathrm{ind}(A)$.

Let A be a central simple F-algebra and $0 \leq d_1 < ... < d_k \leq \deg(A)$ be a sequence of integers. A convenient way to define the variety of flags of right ideals of reduced dimension d_1,..., d_k in A is to use the language of functor of points. For any R in F-$\mathfrak{alg}$, the set of R-points $\mathrm{Mor}_F\left(\mathrm{Spec}(R), X(d_1,...,d_k;A)\right)$ consists of the sequences $(I_1,...,I_k)$ of right ideals of the Azumaya R-algebra $A \otimes_F R$ such that $I_1 \subset ... \subset I_k$, the injection of A_R modules $I_s \to A_R$ splits and the rank of the R-module I_s is equal to $d_s \cdot \deg(A)$ for any $1 \leq s \leq k$. For any morphism $R \to S$ of F-algebras the corresponding map from R-points to S-points is given by $(I_1,...,I_k) \mapsto (I_1 \otimes_R S,...,I_k \otimes_R S)$. Two important particular cases of varieties of flags of right ideals are the classical Severi-Brauer variety $X(1;A)$, and the generalized Severi Brauer varieties $X(i;A)$. If G is an algebraic group and X a projective G-homogeneous F-variety, we say that X is *isotropic* if X has a zero-cycle of degree coprime to p, and X is *anisotropic* if X is not isotropic. If $X = X(d_1,...,d_k;A)$ is a variety of flags of right ideals, X is isotropic if and only if $v_p(\gcd(d_1,...,d_k)) \geq v_p(\mathrm{ind}(A))$. Note that if the Schur index of A is a power of p, X is isotropic if and only if X has a rational point.

3 The theory upper motives

Our basic references for the definitions and the main properties of Chow groups with coefficients and the category $\mathrm{CM}(F;\Lambda)$ of Grothendieck Chow motives with coefficients in a commutative ring Λ are [2] and [5]. In the sequel G will be a semisimple affine adjoint algebraic group of inner type, X a projective G-homogeneous F-variety and Λ will be assumed to be a finite and connected ring. By [3] (see also [8]) the motive of X decomposes in a unique way (up to isomorphism) as a direct sum of indecomposable motives under these assumptions. Among all the indecomposable direct summands in the complete motivic decomposition of X, the (uniquely determined up to isomorphism) indecomposable direct summand M such that the 0-codimensional Chow group of M is non-zero is the *upper motive* of X.

Upper motives are essential : any indecomposable direct summand in the complete motivic decomposition of X is the upper motive of another projective G-homogeneous F-variety by [8, Theorem 3.5]. This structural result implies that the study of the motivic decomposition of a projective G-homogeneous F-variety X is reduced to the case $\Lambda = \mathbb{F}_p$. Indeed by [16, Corollary 2.6] the complete motivic decomposition of X with coefficients in Λ remains the same when passing to the residue field of Λ, and is also the same as if the ring of coefficients is $\mathbb{F}_p$ by [4, Theorem 2.1], where p is the characteristic of the residue field of Λ. These results motivate the study of the set $\mathfrak{X}_G$ of *upper p-motives* of the algebraic group G, which consists of the isomorphism classes of upper motives of projective G-homogeneous F-varieties in $\mathrm{CM}(F;\mathbb{F}_p)$. Furthermore the dimension of the upper motive of X in $\mathrm{CM}(F;\mathbb{F}_p)$ is equal to the canonical p-dimension of X by [6, Theorem 5.1], hence upper motives encode important information on the underlying varieties. Upper motives also have good properties : the upper motives of two projective G-homogeneous F-varieties X and X' in $\mathrm{CM}(F;\mathbb{F})$ are isomorphic if and only if both $X_{F(X')}$ and $X'_{F(X)}$ are isotropic by [8, Corollary 2.15]. The variety X is isotropic if and only if the upper motive of X is isomorphic to the *Tate motive* (that is to say the motive of $\mathrm{Spec}(F)$) and this is why we focus in this note on the case of anisotropic varieties of flags of right ideals.

If G is absolutely simple adjoint of inner type A_n, G is isomorphic to $\mathrm{PGL}_1(A)$, where A is a central simple F-algebra of degree $n+1$. Any projective G-homogeneous F-variety is then isomorphic to a variety $X(d_1, ..., d_k; A)$ of flags of right ideals in A (see [10]) thus theorem 1 classifies upper motives of absolutely simple affine adjoint algebraic groups of inner type A_n. In the particular case of classical Severi-Brauer varieties theorem 1 corresponds to [1, Theorem 9.3], since for any field extension E/F a central simple F-algebra split over E if and only if the Severi-Brauer variety $SB(1, A_E)$ has a rational point.

4 Main results

Let D be a central division F-algebra of degree p^n. For any $0 \leq k \leq n$, $M_{k,D}$ will denote the upper indecomposable direct summand of $X(p^k; D)$ in $\mathrm{CM}(F;\mathbb{F})$. If D' is another central division F-algebra of degree p^n and j satisfies $1 \leq j \leq p^n$,

we denote the integer $\frac{p^k}{\gcd(j,p^k)} \cdot \mathrm{ind}(D \otimes D'^{-j})$ by $\mu_{k,j}^{D,D'}$. In the sequel the following index reduction formula (see [10, p. 565]) will be of constant use :

$$\mathrm{ind}(D_{F(X(p^k;D'))}) = \gcd_{1 \le j \le p^n} \mu_{k,j}^{D,D'} = \min_{1 \le j \le p^n} \mu_{k,j}^{D,D'}$$

Proposition 2. Let D and D' be two central division F-algebras of degree p^n. Assume that $\exp(D) \ge \exp(D')$ and that $X(p^k;D)_{F(X(p^k;D'))}$ is isotropic for some integer $0 \le k < n$. If $\mathrm{ind}(D_{F(X(k;D'))}) = \mu_{k,j_0}^{D,D'}$, j_0 is coprime to p.

Proof. Suppose that p divides j_0 and $\mathrm{ind}(D_{F(X(k;D'))}) = \mu_{k,j_0}^{D,D'}$. By assumption $X(k;D)_{F(X(k;D'))}$ has a rational point, hence the integer $\mu_{k,j_0}^{D,D'}$ divides p^k by [9, Proposition 1.17] and $\mathrm{ind}(D \otimes D'^{-j_0}) \mid \gcd(j_0, p^k)$. Since p divides j_0, $\exp(D'^{-j_0}) < \exp(D')$, therefore $\exp(D'^{-j_0}) < \exp(D)$ and $\exp(D) = \exp(D \otimes D'^{-j_0})$. It follows that $\exp(D)$ divides j_0, thus $\exp(D')$ also divides j_0. The central simple F-algebra D'^{j_0} is therefore split and $D \otimes D'^{-j_0}$ is Brauer-equivalent to D so that $\mathrm{ind}(D)$ divides p^k, a contradiction. □

Theorem 3. Let $\mathbb{F}$ be a finite field of characteristic p and D, D' be two central division F-algebras of degree p^n. The following assertions are equivalent :
1) for some integer $0 \le l < n$, $M_{l,D}$ and $M_{l,D'}$ are isomorphic in $\mathrm{CM}(F;\mathbb{F})$;
2) the classes of D and D' generate the same subgroup of $\mathrm{Br}(F)$;
3) for any $0 \le l < n$, $M_{l,D}$ is isomorphic to $M_{l,D'}$ in $\mathrm{CM}(F;\mathbb{F})$.

Proof. We first show that 1) $\Rightarrow$ 2). We may replace D by D' and thus assume that $\exp(D)$ is greater than $\exp(D')$. Since $M_{l,D}$ is isomorphic to $M_{l,D'}$, there is an integer j_0 coprime to p such that the Schur index of $D \otimes D'^{-j_0}$ is equal to 1 by [9, Proposition 1.17] and proposition 2, hence $D \otimes D'^{-j_0}$ is split and the class of D is equal to the class of D'^{j_0} in $\mathrm{Br}(F)$. Furthermore since j_0 is coprime to p the class of D in $\mathrm{Br}(F)$ is also a generator of the subgroup of $\mathrm{Br}(F)$ generated by $[D']$. Now statement 2) $\Rightarrow$ 3) : if $[D]$ and $[D']$ generate the same group in $\mathrm{Br}(F)$, $\mathrm{ind}(D_E) = \mathrm{ind}(D'_E)$ for any field extension E/F. Given an integer $0 \le l < n$, since $X(p^l;D)$ has a rational point over $F(X(p^l;D))$, $\mathrm{ind}(D'_{F(X(p^l;D))}) = \mathrm{ind}(D_{F(X(p^l;D))})$ divides p^l. The variety $X(p^l;D')$ then also has a rational point over $F(X(p^l;D))$ by [9, Proposition 1.17]. The same procedure replacing D by D' shows that $X(p^l;D)$ has a rational point over $F(X(p^l;D'))$, hence $M_{l,D}$ is isomorphic to $M_{l,D'}$. Finally 3) $\Rightarrow$ 1) is obvious. □

Corollary 4. Let D and D' be two central division F-algebras of degree p^n and $p^{n'}$. The upper summands $M_{k,D}$ and $M_{l,D'}$ are isomorphic for some integers $0 \le k < n$ and $0 \le l < n'$ if and only if $k = l$ and the classes of D and D' generate the same subgroup of $\mathrm{Br}(F)$.

Proof. Since by [8, Theorem 4.1] the generalized Severi-Brauer varieties $X(p^k;D)$ and $X(p^l;D')$ are p-incompressible, if $M_{k,D}$ and $M_{l,D'}$ are isomorphic, the dimension of $X(p^k;D)$ (which is $p^k(p^n - p^k)$) is equal to the dimension of $X(p^l;D')$. The equality $p^k(p^n - p^k) = p^l(p^{n'} - p^l)$ implies that $k = l$, $n = n'$ and it remains to apply theorem 3. The converse is clear by theorem 3. □

Proof of theorem 1. Set $X = X(d_1, ..., d_k; A)$, $Y = X(d_1, ..., d_k; A')$, and also $u = v_p(\gcd(d_1, ..., d_k))$ and $v = v_p(\gcd(d'_1, .., d'_s))$. If D and D' are two central division F-algebras Brauer-equivalent to A_p and A'_p, the upper indecomposable direct summand of X (resp. of Y) is isomorphic to $M_{u,D}$ (resp. to $M_{v,D'}$) by [8, Theorem 3.8]. By corollary 4 these summands are isomorphic if and only if $u = v$ (since X and Y are anisotropic) and the classes of A_p and A'_p generate the same subgroup of $\mathrm{Br}(F)$. $\square$

Theorem 5. Let G and G' be two absolutely simple affine adjoint algebraic groups of inner type A_n and $A_{n'}$. Then either $\mathfrak{X}_G \cap \mathfrak{X}_{G'}$ is reduced to the class of the Tate motive or $\mathfrak{X}_G = \mathfrak{X}_{G'}$.

Proof. If $\mathfrak{X}_{\mathrm{PGL}_1(A)} \cap \mathfrak{X}_{\mathrm{PGL}_1(A')}$ is not reduced to the class of the Tate motive, there are two anisotropic varieties of flags of right ideals $X = X(d_1, ..., d_k; A)$ and $Y = X(d'_1, ..., d'_s; A')$ whose upper motives are isomorphic. By theorem 1 this implies that the upper p-motive of any anisotropic $\mathrm{PGL}_1(A)$-homogeneous F-variety $X(d_1, ..., d_s; A)$ is isomorphic to, say, the upper p-motive of $X(d_1, ..., d_s; A')$. $\square$

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References

[1] Amitsur, S. A. *Generic splitting fields of central simple algebras*, Ann. of Math. 62, 8-43, 1955.

[2] André, Y. *Une introduction aux motifs : (Motifs purs, motifs mixtes, périodes)*, Société Mathématique de France, Panoramas et synthèses 17, 2004.

[3] Chernousov, V., Merkurjev, A. *Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem*, Transformation Groups 11, 371-386, 2006.

[4] De Clercq, C. *Motivic decompositions of projective homogeneous varieties and change of coefficients.*, C.R. Math. Acad. Sci. Paris 348 17-18, 955-958, 2010.

[5] Elman, R., Karpenko, N., Merkurjev, A. *The Algebraic and Geometric Theory of Quadratic Forms*, American Mathematical Society, Providence, 2008.

[6] Karpenko, N. *Canonical dimension*, Proceedings of the ICM 2010, vol. II, 146–161.

[7] Karpenko, N. *On the first Witt index of quadratic forms.*, Invent. Math. 153, no. 2, 455-462, 2003.

[8] Karpenko, N. *Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties*, Linear Algebraic Groups and Related Structures (preprint server) 333, 2009.

[9] Knus, M.-A., Merkurjev, A., Rost, M., Tignol, J.-P. *The book of involutions*, AMS Colloquium Publications, Vol. 44, 1998.

[10] Merkurjev, A., Panin, A., Wadsworth, A. *Index reduction formulas for twisted flag varieties, I*, Journal K-theory 10, 517-596, 1996.

[11] Petrov, V., Semenov, N. *Higher Tits indices of algebraic groups*, Preprint, 2007.

[12] Petrov, V., Semenov, N., K.Zainoulline *J-invariant of linear algebraic groups*, Ann. Sci. Éc. Norm. Sup. 41, 1023-1053, 2008.

[13] Vishik, A. *Motives of quadrics with applications to the theory of quadratic forms*, Lect. Notes in Math. 1835, Proceedings of the Summer School "Geometric Methods in the Algebraic Theory of Quadratic Forms, Lens 2000", 25-101, 2004.

[14] Vishik, A. *Excellent connections in the motives of quadrics*, Annales Scientifiques de L'ENS, in press, 2010.

[15] Vishik, A. *Fields of u-invariant* $2^r + 1$, "Algebra, Arithmetic and Geometry - In Honor of Yu.I.Manin", Birkhauser, 661-685, 2010.

[16] Vishik, A., Yagita, N. *Algebraic cobordism of a Pfister quadric*, J. Lond. Math. Soc. (2) 76, 3, 586-604, 2007.